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LETTER TO THE EDITOR

Photon time of arrival, time between consecutive photons and the moment generating function

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Abstract. The probability densities of the time of arrival of photoelectrons and the time interval between consecutive photoelectrons are respectively proportional to the first and second time derivatives of the moment generating function. Expressions for these densities are given for several classes of optical fields.

The statistical properties of light can be studied by observing the statistics of detected photoelectrons. An important technique for studying the statistics of photoelectrons is to observe the probability distribution of time intervals (Arecchi *et al* 1966, Morgan and Mandel 1966, Glauber 1967). Time interval measurements have been used for the estimation and detection of optical signals in optical communications (Davidson and Amoss 1973). Also, because the distribution of the time interval between consecutive photons is the simplest measurement that depends on the second-order field-correlation function (Present and Scarl 1972), it can be used for the spectral analysis of optical signals (Mandel 1963, Wolf 1965).

Two time intervals are of interest: (i) the time of arrival T (sometimes called the residual waiting time, or the forward recurrence time, Cox and Lewis 1968), which is the time interval between an arbitrary time and the first photoelectron registered after it; (ii) the time interval τ between consecutive photoelectrons (also called the lifetime). If the photoelectron statistics is purely Poisson then T and τ have the same probability distribution which is exponential (Cox and Lewis 1968).

We have previously shown (Saleh 1973) that the probability distribution $f_1(T)$ of the time of arrival T, is related to the optical field's moment generating function (MGF) by a very simple rule (equation (9) below). In this letter, it is shown that the probability distribution $f_2(\tau)$ of the time τ between consecutive photoelectrons is also related to the MGF by another simple rule (equation (10) below). These rules, which are valid for any stationary light field, are used to derive expressions for $f_1(T)$ and $f_2(\tau)$ for several important fields whose MGF have been found by other authors. Gaussian light with lorentzian or rectangular spectrum, and mixtures of coherent and gaussian light are treated.

Let $I(\boldsymbol{r},t)$ represent the light intensity at position $\boldsymbol{r}$ and time t (measured in photon numbers per second), and let $U(t) = \int_A I(\boldsymbol{r},t)\, d^2\boldsymbol{r}$ be its integration over the detector area A. For stationary light the expectation value of $U(t)$ is independent of time and equals the average number of counts per second $\langle U(t)\rangle = \bar{n}$. The moment generating function (MGF) of $U(t)$ is defined as

$$Q(s,T) = \left\langle \exp\left(s\int_0^T U(t)\,\mathrm{d}t\right)\right\rangle. \tag{1}$$

It is known that conditioned on a certain realization of the stochastic process $U(t)$, the photoelectron statistics is Poisson (Jakeman and Pike 1969). From the well known properties of a Poisson process we can write our time interval probability distributions as

$$f_1(T) = \left\langle U(T)\exp\left(-\int_0^T U(t)\,\mathrm{d}t\right)\right\rangle, \tag{2}$$

and

$$f_2(\tau) = \frac{\langle U(0)\,U(\tau)\exp(-\int_0^\tau U(t)\,\mathrm{d}t)\rangle}{\langle U(t)\rangle}. \tag{3}$$

We are interested in using (1), (2) and (3) to find a relation between the probability distributions and the MGF. It is useful to start by considering the cumulative probability distributions defined as

$$F_1(T) = \int_0^T f_1(x)\,\mathrm{d}x, \tag{4}$$

$$F_2(\tau) = \int_0^\tau f_2(x)\,\mathrm{d}x. \tag{5}$$

By substituting (2) in (4) we get

$$F_1(T) = \left\langle \int_0^T U(x)\exp\left(-\int_0^x U(t)\,\mathrm{d}t\right)\mathrm{d}x\right\rangle = 1-\left\langle \exp\left(-\int_0^T U(t)\,\mathrm{d}t\right)\right\rangle,$$

from which

$$F_1(T) = 1-Q(1,T). \tag{6}$$

Similarly, we use (3) in (5) and get

$$F_2(\tau) = \frac{\langle U(0)\int_0^\tau U(x)\exp(-\int_0^x U(t)\,\mathrm{d}t)\,\mathrm{d}x\rangle}{\bar{n}}$$

$$= 1-\frac{\langle U(0)\exp(-\int_0^T U(t)\,\mathrm{d}t)\rangle}{\bar{n}}. \tag{7}$$

To simplify this further, we write

$$Q(s,T) = Q(-s,-T) = \left\langle \exp\left(s\int_0^{-T} U(t)\,\mathrm{d}t\right)\right\rangle,$$

and take its derivative

$$\frac{\partial}{\partial T}Q(s,T) = -\frac{\partial}{\partial(-T)}Q(-s,-T) = -\left\langle sU(-T)\exp\left(s\int_0^{-T} U(t)\,\mathrm{d}t\right)\right\rangle,$$

then using the stationary property of $U(t)$ we get

$$\frac{\partial}{\partial T}Q(s,T) = -\left\langle sU(0)\exp\left(s\int_T^0 U(t)\,\mathrm{d}t\right)\right\rangle,$$

which when substituted in (7) gives

$$F_2(\tau) = 1+\frac{1}{\bar{n}}\frac{\partial}{\partial\tau}Q(1,\tau). \tag{8}$$

Since the probability densities $f_1(T)$ and $f_2(\tau)$ are the derivatives of $F_1(T)$ and $F_2(\tau)$, then

$$f_1(T) = -\frac{\partial}{\partial T}Q(1,T), \tag{9}$$

and

$$f_2(\tau) = \frac{1}{\bar{n}}\frac{\partial^2}{\partial\tau^2}Q(1,\tau) = -\frac{1}{\bar{n}}\frac{\partial}{\partial\tau}f_1(\tau). \tag{10}$$

Equation (10) is the main result of this letter. Expressions for the moment generating function have been found for most important optical fields (Jakeman and Pike 1969) and it appears very simple to use any of (6), (8), (9) or (10) and find the desired time-interval distribution. In the following, we give a list of expressions of $f_1(T)$ and $f_2(\tau)$ for some of the important classes of optical fields.

(i) *Coherent light.* The MGF of a coherent field is (Glauber 1967)

$$Q(s,T) = \exp(-s\bar{n}T),$$

hence,

$$f_1(T) = \bar{n}\exp(-\bar{n}T),$$
$$f_2(\tau) = \bar{n}\exp(-\bar{n}\tau). \tag{11}$$

Note that, $f_1(T) = f_2(T)$ is the exponential distribution characterizing a purely Poisson photoelectron process.

(ii) *Gaussian light with long coherence time.* A gaussian optical field whose coherence time Γ^{-1} is much longer than $(1/\bar{n})$, the average time interval, has a MGF

$$Q(s,T) \simeq (1+s\bar{n}T)^{-1}.$$

In this case

$$f_1(T) = \bar{n}(1+\bar{n}T)^{-2}, \tag{12}$$

and

$$f_2(\tau) = 2\bar{n}(1+\bar{n}\tau)^{-3}, \tag{13}$$

which have been obtained before (Glauber 1967).

(iii) *Gaussian–lorentzian light.*

$$Q(s,T) = \exp(\gamma\bar{n}T)[\cosh(\alpha(s)\bar{n}T)+\beta(s)\sinh(\alpha(s)\bar{n}T)]^{-1}$$

where

$$\gamma = \frac{\Gamma}{\bar{n}}$$
$$\alpha(s) = (\gamma^2+2\gamma s)^{1/2},$$
$$\beta(s) = \frac{1}{2}\left(\frac{\gamma}{\alpha(s)}+\frac{\alpha(s)}{\gamma}\right),$$

and Γ is the bandwidth. This MGF gives the probability densities,

$$f_1(T) = \frac{\bar{n}Q(1,T)2\gamma A}{B}, \tag{14}$$

$$f_2(\tau) = \frac{\bar{n}Q(1,\tau)4\gamma^2(A^2+4\alpha^2)}{B^2}, \tag{15}$$

where

$$A = (\gamma+\alpha)\exp(\alpha\bar{n}T)-(\gamma-\alpha)\exp(-\alpha\bar{n}T),$$
$$B = (\gamma+\alpha)^2\exp(\alpha\bar{n}T)-(\gamma-\alpha)^2\exp(-\alpha\bar{n}T),$$

and $\alpha = \alpha(1)$. In the limit of long coherence time ($\gamma = 0$), (14) and (15) give back (12) and (13) above.

(iv) *Gaussian–rectangular light.* Gaussian light with a rectangular shaped spectrum having a bandwidth Γ has MGF approximately given by (Karp and Clark 1971),

$$Q(s,T) \simeq (1+s\bar{n}T)^{-(1+2\gamma\bar{n}T)}$$

where $\gamma = \Gamma/\bar{n}$. This gives the probability densities

$$f_1(T) = \bar{n}Q(1,T)\left(\frac{1+2\gamma\bar{n}T}{1+\bar{n}T}+2\gamma\ln(1+\bar{n}T)\right), \tag{16}$$

$$f_2(\tau) = \bar{n}Q(1,\tau)\left[\left(\frac{1+2\gamma\bar{n}\tau}{1+\bar{n}\tau}+2\gamma\ln(1+\bar{n}\tau)\right)^2-\frac{2\gamma(\bar{n}\tau+2)-1}{(1+\bar{n}\tau)^2}\right]. \tag{17}$$

In the limit $\gamma \to 0$, (12) and (13) are reproduced.

(v) *A mixture of coherent and gaussian light* (*long coherence-time limit*).

$$Q(s,T) = \frac{1}{1+s\epsilon\bar{n}T}\left[\exp\left(-\frac{s(1-\epsilon)\bar{n}T}{1+s\epsilon\bar{n}T}\right)\right]$$

where $\bar{n}$ is the average count rate due to the mixed light and ϵ is the ratio of the intensity of the incoherent part to the total intensity. In this case,

$$f_1(T) = \bar{n}Q(1,T)\frac{1+\epsilon^2\bar{n}T}{(1+\bar{n}T)^2}, \tag{18}$$

and

$$f_2(\tau) = \bar{n}Q(1,\tau)\frac{2(1+\epsilon^2\bar{n}\tau)^2-(1-\epsilon)^2}{(1+\epsilon\bar{n}\tau)^4}. \tag{19}$$

The limit $\epsilon = 0$ (coherent light) gives back (11) and the limit $\epsilon = 1$ (incoherent light) recovers (12) and (13). Equations (18) and (19) are valid for light with long coherence time, ie $\gamma \to 0$. This limitation can be removed if we use the already found general expressions of the MGF (Jakeman and Pike 1969, Karp and Clark 1971) which are quite lengthy.

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